

Reproducing kernel Hilbert spaces and random measures *

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Abstract

We present a constructive approach of Guilbart's embedding of signed measures in a reproducing kernel Hilbert space. This enables us to define *signed* random measures as Hilbertian random elements and exploit the nice probabilistic properties of Hilbert space when studying limit theorems for random measures. Strong law of large numbers, central limit theorems, functional central limit theorems are given. As a statistical illustration we propose a test procedure to detect an epidemic change in the distribution of random measures.

Résumé

Nous présentons une approche constructive du plongement, introduit par Guilbart, des mesures signées dans un espace de Hilbert à noyau reproduisant. Cette construction nous permet d'exploiter les bonnes propriétés probabilistes des espaces de Hilbert dans l'étude des théorèmes limites pour les mesures aléatoires : loi des grands nombres, théorème central limite, théorèmes centraux limites fonctionnels. Ces résultats sont illustrés par une procédure de test de détection d'une rupture de type épidémique dans la loi d'une mesure aléatoire.

Keywords : central limit theorem, change point detection, empirical process, epidemic model, functional central limit theorem, mean measure, random measure, reproducing kernel, strong law of large numbers.

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1 R.K.H.S. and metrics on signed measures

In the late seventies, C. Guilbart [4, 5] introduced an embedding into a reproducing kernel Hilbert space (R.K.H.S.) \mathcal{H} of the space \mathcal{M} of signed measures on some topological space \mathfrak{X} . He characterized the inner products on \mathcal{M} inducing the weak topology on the subspace \mathcal{M}^+ of bounded positive measures and established in this setting a Glivenko-Cantelli theorem with applications to estimation and hypothesis testing. In this contribution we present a constructive approach of Guilbart's embedding following [20]. This embedding provides a Hilbertian framework for signed random measures. We shall discuss some applications of this construction to limit theorems for random measures and partial sums processes.

Let \mathfrak{X} be a metric space and let \mathcal{M} denote the space of *signed measures* on the Borel σ -field of \mathfrak{X} . A signed measure μ is the difference of two positive bounded measures. We denote by (μ^+, μ^-) its Hahn-Jordan decomposition and by $|\mu| = \mu^+ + \mu^-$ its total variation measure.

We consider the class of reproducing kernels having the following representation

$$K(x, y) = \int_{\mathbb{U}} r(x, u) \overline{r(y, u)} \rho(du), \quad x, y \in \mathfrak{X}, \quad (1)$$

where ρ is a positive measure on some measurable space $(\mathbb{U}, \mathcal{U})$ and the function $r : \mathfrak{X} \times \mathbb{U} \rightarrow \mathbb{C}$ satisfies

$$\sup_{x \in \mathfrak{X}} \|r(x, \cdot)\|_{L^2(\rho)} < \infty. \quad (2)$$

We denote by \mathcal{H} the reproducing kernel Hilbert space associated with K . It is easily checked (Prop.2 in [20]) that under (2), $r(\cdot, u)$ is μ -integrable over \mathfrak{X} for ρ -almost $u \in \mathbb{U}$. We assume moreover that

$$\text{if } \mu \in \mathcal{M} \text{ and } \int_{\mathfrak{X}} r(x, u) \mu(dx) = 0 \text{ for } \rho\text{-almost } u, \text{ then } \mu = 0. \quad (3)$$

The essential facts about the embeddings of \mathcal{M} into \mathcal{H} and $L^2(\rho)$ are gathered in the following theorem which is proved in [20].

Theorem 1. *Under (1), (2) and (3), the following properties hold.*

- a) *Let E be the closed subspace of $L^2(\rho)$ spanned by $\{r(x, \cdot), x \in \mathfrak{X}\}$. A function $h : \mathfrak{X} \rightarrow \mathbb{C}$ belongs to \mathcal{H} if and only if there is a unique $g \in L^2(\rho)$ such that*

$$h(x) = \int_{\mathbb{U}} g(u) \overline{r(x, u)} \rho(du), \quad x \in \mathfrak{X}. \quad (4)$$

The representation (4) defines a isometry of Hilbert spaces $\Psi : \mathcal{H} \rightarrow E$, $h \mapsto g$.

- b) *K induces an inner product on \mathcal{M} by the formula*

$$\langle \mu, \nu \rangle_K := \int_{\mathfrak{X}^2} K(x, y) \mu \otimes \nu(dx, dy), \quad \mu, \nu \in \mathcal{M}. \quad (5)$$

c) $(\mathcal{M}, \langle \cdot, \cdot \rangle_K)$ is isometric to a dense subspace of \mathcal{H} by

$$\mathcal{J} : \mathcal{M} \rightarrow \mathcal{H}, \quad \mu \mapsto \mathcal{J}_\mu := \int_{\mathfrak{X}} K(x, \cdot) \mu(dx). \quad (6)$$

Moreover we have

$$\langle h, \mathcal{J}_\mu \rangle = \int_{\mathfrak{X}} h d\mu, \quad \langle \mathcal{J}_\mu, h \rangle = \int_{\mathfrak{X}} \bar{h} d\mu, \quad h \in \mathcal{H}, \mu \in \mathcal{M}. \quad (7)$$

d) The isometric embedding $\zeta = \Psi \circ \mathcal{J} : \mu \mapsto \zeta_\mu$ of \mathcal{M} into $L^2(\rho)$ satisfies

$$\zeta_\mu(u) = \int_{\mathfrak{X}} r(x, u) \mu(dx), \quad u \in \mathbb{U}. \quad (8)$$

Let us examine some examples where Theorem 1 applies.

Example 1. Take for ρ the counting measure on $\mathbb{U} = \mathbb{N}$ and define r by $r(x, i) := f_i(x)$, $x \in \mathfrak{X}$, $i \in \mathbb{N}$, where the sequence of functions $f_i : \mathfrak{X} \rightarrow \mathbb{R}$ separates the measures, i.e. the only $\mu \in \mathcal{M}$ such that $\int_{\mathfrak{X}} f_i d\mu = 0$ for all $i \in \mathbb{N}$ is the null measure. To have a bounded kernel we also assume that $\sum_{i \in \mathbb{N}} \|f_i\|_\infty^2 < \infty$. Then

$$K(x, y) = \sum_{i \in \mathbb{N}} f_i(x) f_i(y), \quad x, y \in \mathfrak{X}^2.$$

μ is represented in $\ell^2(\mathbb{N})$ by $\zeta_\mu = (\int_{\mathfrak{X}} f_i d\mu)_{i \in \mathbb{N}}$ and in \mathcal{H} by $\mathcal{J}_\mu = \sum_{i \in \mathbb{N}} (\int_{\mathfrak{X}} f_i d\mu) f_i$. It easily follows from (4) that every f_i belongs to \mathcal{H} .

Example 2. Take $\mathfrak{X} = \mathbb{U} = \mathbb{R}^d$, with $r(x, u) := \exp(i\langle x, u \rangle)$, $x, u \in \mathbb{R}^d$ and choose ρ as a bounded positive measure on \mathbb{R}^d . This gives the continuous *stationary* kernels

$$K(x, y) = \int_{\mathbb{R}^d} \exp(i\langle x - y, u \rangle) \rho(du), \quad x, y \in \mathbb{R}^d.$$

Here $\zeta_\mu(u) = \int_{\mathbb{R}^d} \exp(i\langle x, u \rangle) \mu(dx) =: \hat{\mu}(u)$, is the characteristic function of μ and $\mathcal{J}_\mu(x) = \int_{\mathbb{R}^d} \exp(-i\langle x, u \rangle) \hat{\mu}(u) \rho(du)$. These kernels are used in [20] to study the convergence rate in the CLT.

Example 3. Take $\mathfrak{X} = \mathbb{U} = [0, 1]$, $\rho = \lambda + \delta_1$, where λ is the Lebesgue measure and δ_1 the Dirac mass at point 1. With $r(x, u) := \mathbf{1}_{[x, 1]}(u)$, we obtain $K(x, y) = 2 - \max(x, y)$ and $\zeta_\mu(u) = \mu([0, u])$.

Remark 2. The usual topologies on \mathcal{M} are generated by functionals $f \mapsto \int_{\mathfrak{X}} f d\mu$, $f \in F$, where F is some family of continuous functions defined on \mathfrak{X} . When \mathfrak{X} is locally compact, $F = C(\mathfrak{X})$, the space of all bounded continuous functions on \mathfrak{X} gives the weak topology while restricting to $F = C_0(\mathfrak{X})$ the space of continuous function converging to zero at infinity gives the vague topology.

By convergence to zero at infinity we mean that for every positive ε there is a compact subset A of \mathfrak{X} such that $|f(x)| < \varepsilon$ for every $x \in \mathfrak{X} \setminus A$. In the special case where \mathfrak{X} is compact, $C(\mathfrak{X}) = C_0(\mathfrak{X})$. Endowed with the supremum norm, $C_0(\mathfrak{X})$ is a Banach space with topological dual \mathcal{M} (Riesz's theorem). Now if we choose in Example 1 the f_i 's in $C_0(\mathfrak{X})$, a simple Hahn-Banach argument gives the density of \mathcal{H} in $C_0(\mathfrak{X})$. In this setting, let $(\mu_n)_{n \geq 1}$ a sequence in \mathcal{M} such that $\sup_{n \geq 1} |\mu_n|(\mathfrak{X}) < \infty$. Then weak and strong convergence in \mathcal{H} of J_{μ_n} to J_μ are equivalent to the weak convergence in \mathcal{M} of μ_n to μ .

2 Some limit theorems for random measures

2.1 Random measures

A random measure μ^\bullet is a random element in a set \mathfrak{M} of measures equipped with some σ -field \mathcal{G} , i.e. a measurable mapping

$$\mu^\bullet : (\Omega, \mathcal{F}, P) \longrightarrow (\mathfrak{M}, \mathcal{G}), \quad \omega \mapsto \mu^\omega.$$

Here (Ω, \mathcal{F}, P) is a probability space and the law or distribution of μ^\bullet (under P) is the image measure $P \circ (\mu^\bullet)^{-1}$ on \mathcal{G} . Among the well known examples of random measures let us mention the empirical process $\mu_n^\bullet = n^{-1} \sum_{i=1}^n \delta_{X_i}$, where the X_i 's are i.i.d. random elements in the space \mathfrak{X} and the point processes $\sum_{i=1}^N \delta_{Y_i}$, where N and the Y_i 's are random. In the classical theory, e.g. Kallenberg [7], \mathfrak{X} is locally compact with a countable basis of neighborhoods, \mathfrak{M} is the set of *positive* Radon measures on the Borel σ -field of \mathfrak{X} and \mathfrak{M} is endowed with the Borel σ -field \mathcal{G} of the vague topology. This framework of positive measures is sufficient to the classical study of point processes. But the above setting does not cover the case of signed measures. Still signed measures appear naturally by centering of positive ones[6].

Guilbart's embedding of \mathcal{M} in an R.K.H.S. \mathcal{H} provides the background for a Hilbertian theory of *signed* random measures. This way we can exploit the nice probabilistic properties of Hilbert spaces and obtain useful limit theorems like CLT or FCLT.

From now on, we assume for simplicity that \mathfrak{X} is metric locally compact and that K is as in Example 1 with the f_i 's in $C_0(\mathfrak{X})$. Identifying \mathcal{H} with a completion of \mathcal{M} , we call random measure a random element μ^\bullet in \mathcal{H} such that $P(\mu^\bullet \in \mathcal{M}) = 1$. The *observations* of such a random measure are the random variables $\langle h, \mu^\bullet \rangle_K = \int_{\mathfrak{X}} h \, d\mu^\bullet$, $h \in \mathcal{H}$, accounting (7). Some natural measurability questions raised by our definition of random measures are positively answered in [19]: \mathcal{M} is a Borel subset of \mathcal{H} , $|\mu^\bullet|$ is also a random measure, the $\int_{\mathfrak{X}} f \, d\mu^\bullet$'s, $f \in C_0(\mathfrak{X})$, and $|\mu^\bullet|(\mathfrak{X})$ are random variables.

2.2 Strong law of large numbers

If $\mathbf{E}\|\mu^\bullet\|_K$ is finite, the random measure μ^\bullet is Bochner integrable and $\mathbf{E}\mu^\bullet$ is defined as a deterministic element of \mathcal{H} . Then μ^\bullet is also Pettis integrable,

whence

$$\mathbf{E}\langle h, \mu^\bullet \rangle_K = \langle h, \mathbf{E}\mu^\bullet \rangle_K, \quad h \in \mathcal{H}. \quad (9)$$

The following theorem is an immediate application of the strong law of large numbers in separable Banach spaces, see e.g. [9].

Theorem 3. *Let $\mu_1^\bullet, \dots, \mu_n^\bullet, \dots$ be independent identically distributed copies of μ^\bullet . If $\mathbf{E}\|\mu^\bullet\|_K$ is finite, then*

$$\nu_n^\bullet := \frac{1}{n} \sum_{i=1}^n \mu_i^\bullet \xrightarrow[\text{a.s.}]{\mathcal{H}} \mathbf{E}\mu^\bullet. \quad (10)$$

Conversely, if ν_n^\bullet converges almost surely in \mathcal{H} to some limit ℓ , this limit is deterministic, $\mathbf{E}\|\mu^\bullet\|_K$ is finite and $\ell = \mathbf{E}\mu^\bullet$.

Although ν_n^\bullet is obviously a random measure, it is not clear that the same holds true for its a.s. limit $\mathbf{E}\mu^\bullet$. When $\mathbf{E}\mu^\bullet$ belongs to \mathcal{M} , we call it the *mean measure of μ^\bullet* . In this case, (9) can be recast as

$$\mathbf{E}\langle h, \mu^\bullet \rangle_K = \int_{\mathfrak{X}} h \, d(\mathbf{E}\mu^\bullet), \quad h \in \mathcal{H}. \quad (11)$$

Here is a simple sufficient condition for the existence of the mean measure.

Proposition 4. *The membership of $\mathbf{E}\mu^\bullet$ in \mathcal{M} follows from the finiteness of $\mathbf{E}|\mu^\bullet|(\mathfrak{X})$ if \mathfrak{X} is locally compact, K is continuous on \mathfrak{X}^2 and $K(x, \cdot) \in C_0(\mathfrak{X})$ for every $x \in \mathfrak{X}$.*

The proof (cf. Prop. XI.1.2 in [17]) relies on the characterization of measures in \mathcal{H} by

$$g \in \mathcal{J}(\mathcal{M}) \quad \text{iff} \quad \sup_{f \in \mathcal{H}, \|f\|_\infty \leq 1} |\langle f, g \rangle| < \infty, \quad (12)$$

using the fact that when finite, the supremum in (12) equals $|\mu|(\mathfrak{X})$, where $\mu := \mathcal{J}^{-1}(g)$, together with the elementary estimate

$$\|\mu\|_K \leq \left(\sup_{\mathfrak{X}^2} K \right)^{1/2} |\mu|(\mathfrak{X}), \quad \mu \in \mathcal{M}. \quad (13)$$

Corollary 5. *If $\mathbf{E}|\mu^\bullet|(\mathfrak{X}) < \infty$, let μ be the mean measure of μ^\bullet . Then the a.s. convergence of ν_n^\bullet to μ holds both in \mathcal{H} and in the weak topology on \mathcal{M} .*

The a.s. convergence in \mathcal{H} follows obviously from Theorem 3 applying (13) to μ^\bullet . By Remark 2, (10) implies the a.s. weak convergence in \mathcal{M} of ν_n^\bullet to μ provided that $\sup_{n \geq 1} |\nu_n^\bullet|(\mathfrak{X}) < \infty$. This uniform boundedness follows from the estimate $|\nu_n^\bullet|(\mathfrak{X}) \leq n^{-1} \sum_{i=1}^n |\mu_i^\bullet|(\mathfrak{X})$ and of the a.s. convergence of this upper bound to $\mathbf{E}|\mu^\bullet|(\mathfrak{X})$ by the strong law of large numbers applied to the i.i.d. random variables $|\mu_i^\bullet|(\mathfrak{X})$.

2.3 Central limit theorem for i.i.d. summands

In any separable Hilbert space H , the central limit theorem for a sum of i.i.d. random elements is equivalent to the square integrability of the summands. This nice property does not extend to general Banach spaces, because the CLT is deeply connected to the geometry of the space [9]. A square integrable random element X in H is always *pregaussian*, i.e. there is a Gaussian random element in H with the same covariance structure as X .

Theorem 6. *Let $\mu_1^\bullet, \dots, \mu_n^\bullet, \dots$ be i.i.d. copies of μ^\bullet . If $\mathbf{E}\|\mu^\bullet\|_K^2 < \infty$, then*

$$S_n^* := \frac{1}{\sqrt{n}} \sum_{i=1}^n (\mu_i^\bullet - \mathbf{E}\mu^\bullet) \xrightarrow[\text{in law}]{\mathcal{H}} \gamma^\bullet, \quad (14)$$

where γ^\bullet is a Gaussian random element in \mathcal{H} with $\mathbf{E}\gamma^\bullet = 0$ and covariance given by

$$\text{Cov}(\gamma^\bullet)(f, g) = \mathbf{E} \left(\int_{\mathfrak{X}} f \, d\mu^\bullet \int_{\mathfrak{X}} g \, d\mu^\bullet \right) - \left(\mathbf{E} \int_{\mathfrak{X}} f \, d\mu^\bullet \right) \left(\mathbf{E} \int_{\mathfrak{X}} g \, d\mu^\bullet \right), \quad (15)$$

for every $f, g \in \mathcal{H}$.

Conversely, if S_n^* converges in law in \mathcal{H} , its limit is Gaussian and $\mathbf{E}\|\mu^\bullet\|_K^2 < \infty$.

Corollary 7. *If \mathfrak{X} is locally compact and $\mathbf{E}|\mu^\bullet|(\mathfrak{X})^2 < \infty$, then both μ^\bullet and $\mu^\bullet \otimes \mu^\bullet$ have mean measures, μ and ν and (14) holds. In this case, (15) can be recast as*

$$\text{Cov}(\gamma^\bullet)(f, g) = \int_{\mathfrak{X}^2} f \otimes g \, d\nu - \left(\int_{\mathfrak{X}} f \, d\mu \right) \left(\int_{\mathfrak{X}} g \, d\mu \right).$$

Example 4 (CLT for empirical measure). Let X be a random element $(\Omega, \mathcal{F}, P) \rightarrow (\mathfrak{X}, \mathcal{B}_{\mathfrak{X}})$ with unknown distribution $\mu = P \circ X^{-1}$. Denote by X_1, \dots, X_n , i.i.d. copies of X and put $\mu_i^\bullet := \delta_{X_i}$, $i = 1, \dots, n$. Then $n^{-1} \sum_{i=1}^n \delta_{X_i}$ is the *empirical measure* associated to the sample X_1, \dots, X_n . The CLT in \mathcal{H} for the empirical measure was obtained by Berline [2] by a direct approach. It can also be seen as a special case of Corollary 7. Indeed here $\mu^\bullet = \delta_X$, so $|\mu^\bullet|(\mathfrak{X}) = 1$, $\mathbf{E}\mu^\bullet = \mu = P \circ X^{-1}$ and $\mathbf{E}(\mu^\bullet \otimes \mu^\bullet) =: \nu$ is the image measure of $P \circ X^{-1}$ by the mapping $x \mapsto (x, x)$. Hence

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n \delta_{X_i} - \mu \right) \xrightarrow[\text{in law}]{\mathcal{H}} \gamma^\bullet,$$

where the covariance of the Gaussian centered random element γ^\bullet is given by

$$\text{Cov}(\gamma^\bullet)(f, g) = \int_{\mathfrak{X}} f(x)g(x) \, d\mu(x) - \left(\int_{\mathfrak{X}} f \, d\mu \right) \left(\int_{\mathfrak{X}} g \, d\mu \right).$$

2.4 CLT for Donsker random measure and FCLT in $L^2[0, 1]$

It is also possible to obtain central limit theorems for sums of non i.i.d. random measures, like the Donsker random measure

$$\nu_n^\bullet := \frac{1}{s_n} \sum_{i=1}^n X_i \delta_{\frac{i}{n}}, \quad n \geq 1, \quad (16)$$

where the X_i 's are mean zero random variables, possibly dependent, with $s_n^2 := \mathbf{E}S_n^2$ and $S_n = \sum_{i=1}^n X_i$. One application of such CLT is a functional central limit theorem (FCLT) in $L^2[0, 1]$ for the partial sums processes

$$W_n(t) := s_n^{-1} S_{[nt]}, \quad t \in [0, 1]. \quad (17)$$

This application was suggested by P. Jacob to P.E. Oliveira and the author. The weak convergence of W_n is classically studied in the Skorohod space $D(0, 1)$ which is continuously embedded in $L^2[0, 1]$. As many test statistics are functionals continuous in $L^2[0, 1]$ sense of W_n or of the empirical process, see [11] and [13], the weaker topological framework of $L^2[0, 1]$ has its own interest. This way we can hope to relax the assumptions on the dependence structure of the underlying variables X_i 's. We just sketch here the method and refer to [10, 11] for more precise results.

Let us choose $\mathfrak{X} = [0, 1]$ with the kernel of Example 3. Then

$$\zeta_{\nu_n^\bullet}(t) = \nu_n^\bullet([0, t]) = s_n^{-1} S_{[nt]} = W_n(t), \quad t \in [0, 1]. \quad (18)$$

Hence by isometry between the Hilbert spaces \mathcal{H} and $L^2[0, 1]$,

$$\nu_n^\bullet \xrightarrow[\text{in law}]{\mathcal{H}} \gamma^\bullet \iff W_n \xrightarrow[\text{in law}]{L^2[0,1]} W, \quad (19)$$

where under mild assumptions, the limiting process W is identified as a Brownian motion by a simple covariance computation. Now the relevant CLT for ν_n^\bullet may be established by checking the following conditions.

- a) The inner products $\langle h, \nu_n^\bullet \rangle_K$ converge in law to $\langle h, \gamma^\bullet \rangle_K$ for any fixed $h \in \mathcal{H}$.
- b) The sequence $(\nu_n^\bullet)_{n \geq 1}$ is tight in \mathcal{H} , i.e. for any positive ε , there is a compact subset C_ε of \mathcal{H} such that $\inf_{n \geq 1} P(\nu_n^\bullet \in C_\varepsilon) \geq 1 - \varepsilon$.

The first condition reduces to a CLT in \mathbb{R} for triangular arrays because

$$\langle h, \nu_n^\bullet \rangle_K = \frac{1}{s_n} \sum_{i=1}^n X_i \langle h, \delta_{\frac{i}{n}} \rangle_K = \frac{1}{s_n} \sum_{i=1}^n h\left(\frac{i}{n}\right) X_i. \quad (20)$$

By an adaptation of a classical Prohorov's result (Th.1.13 in [14]), sufficient conditions for the tightness of $(\nu_n^\bullet)_{n \geq 1}$ are

$$\sup_{n \geq 1} \mathbf{E} \|\nu_n^\bullet\|_K^2 < \infty, \quad (21)$$

$$\lim_{n \rightarrow \infty} \sup_{n \geq 1} \mathbf{E} \sum_{i \geq N} |\langle f_i, \nu_n^\bullet \rangle_K|^2 = 0, \quad (22)$$

for some Hilbertian basis $(f_i)_{i \in \mathbb{N}}$ of \mathcal{H} . Concerning (21) which does not come from Th.1.13 in [14], see the remark after Theorem 5 in [21].

Now the heart of the matter is in the following elementary estimate.

$$\begin{aligned} \mathbf{E} \sum_{i \geq N} |\langle f_i, \nu_n^\bullet \rangle_K|^2 &= \sum_{i \geq N} \mathbf{E} \left(\int f_i d\nu_n^\bullet \right)^2 \\ &= \sum_{i \geq N} \frac{1}{s_n^2} \sum_{j,k=1}^n \mathbf{E}(X_j X_k) f_i\left(\frac{j}{n}\right) f_i\left(\frac{k}{n}\right) \\ &\leq \left(\frac{1}{s_n^2} \sum_{j,k=1}^n |\mathbf{E}(X_j X_k)| \right) \sup_{x \in [0,1]} \sum_{i \geq N} f_i(x)^2. \quad (23) \end{aligned}$$

The first factor in (23) may be bounded uniformly in n , subject to good covariance estimates for the X_j 's. The second factor goes to zero due to Dini's theorem (the f_i 's being continuous like any element of \mathcal{H}). Moreover (21) follows obviously from (23) with $N = 0$ in the same setting.

To sum up, the FCLT in $L^2[0, 1]$ for the partial sums process W_n based on the dependent sequence $(X_j)_{j \geq 1}$ is obtained under an estimate $\sum_{j,k=1}^n |\mathbf{E}(X_j X_k)| = O(s_n^2)$ and a onedimensional CLT for the triangular arrays (20).

2.5 Functional central limit theorems

The vector space structure of \mathcal{M} enables us to define polygonal lines of a partial sums process of random measures in view to extend the classical FCLT for random variables. More precisely, let μ^\bullet be a signed random measure and the μ_i^\bullet 's be i.i.d. copies of μ^\bullet . We denote by ξ_n^\bullet the \mathcal{M} -valued stochastic process indexed by $[0, 1]$, whose paths are polygonal lines with vertices $(k/n, n^{-1/2} S_k)$, $k = 0, 1, \dots, n$, $S_k := \mu_1^\bullet + \dots + \mu_k^\bullet$.

Combining Theorem 6 with Kuelbs FCLT [8], we immediately obtain the FCLT for ξ_n^\bullet in the space $\mathcal{C}([0, 1], \mathcal{H})$ of continuous functions $[0, 1] \rightarrow \mathcal{H}$.

Theorem 8. *The following statements are equivalent.*

- a) $\mathbf{E} \|\mu^\bullet\|_K^2 < \infty$ and $\mathbf{E} \mu^\bullet = 0$,
- b) ξ_n^\bullet converges in law in $\mathcal{C}([0, 1], \mathcal{H})$ to some \mathcal{H} -valued Brownian motion W , i.e. a Gaussian process with independent increments such that $W(t) - W(s)$ has the same distribution as $|t - s|^{1/2} \gamma^\bullet$, where γ^\bullet is a Gaussian random element in \mathcal{H} with null expectation and same covariance structure as μ^\bullet .

As the paths of ξ_n^\bullet are Lipschitz \mathcal{H} -valued functions, it is natural to look for a stronger topological framework than $\mathcal{C}([0, 1], \mathcal{H})$ for the FCLT. A natural border in this area is provided by the modulus of uniform continuity of the limiting process, $\omega(W, u) := \sup_{0 \leq t-s \leq u} \|W(t) - W(s)\|_{\mathcal{H}}$, which by a simple projection argument and Lévy's well known result cannot be better than $u^{1/2} \ln(1/u)$.

Introduce the separable Hölder spaces $H_\rho^\alpha([0, 1], \mathcal{H})$ of functions $f : [0, 1] \rightarrow \mathcal{H}$, such that

$$\|f\|_\rho := \|f(0)\|_{\mathcal{H}} + \omega_\rho(f, 1) < \infty \quad \text{and} \quad \lim_{u \rightarrow 0} \omega_\rho(f, u) = 0,$$

where

$$\omega_\rho(f, u) := \sup_{0 < t-s \leq u} \frac{\|f(t) - f(s)\|_{\mathcal{H}}}{\rho(t-s)}.$$

We assume moreover that the weight functions ρ are of the form $\rho(u) = u^\alpha L(1/u)$, $0 < \alpha < 1/2$, where L is continuous normalized slowly varying at infinity. The $H_\rho^\alpha([0, 1], \mathcal{H})$ weak convergence of ξ_n^\bullet to W requires stronger integrability of μ^\bullet than Condition a) in Theorem 8. Combining Theorem 6 with the Hölderian FCLT in [15], leads to the FCLT for ξ_n^\bullet in the space $H_\rho^\alpha([0, 1], \mathcal{H})$.

Theorem 9. *Assume that there is a $\beta > 1/2$ such that*

$$t^{1/2} \rho(1/t) \ln^{-\beta}(t) \text{ is non decreasing on some } [a, \infty). \quad (24)$$

Then the following statements are equivalent.

a) $\mathbf{E}\mu^\bullet = 0$ and

$$\text{for every } A > 0, \quad \lim_{t \rightarrow \infty} tP(\|\mu^\bullet\|_K \geq At^{1/2} \rho(1/t)) = 0. \quad (25)$$

b) ξ_n^\bullet converges in law in $H_\rho^\alpha([0, 1], \mathcal{H})$ to the \mathcal{H} -valued Brownian motion W of Th. 8.

When $\alpha < 1/2$, Condition (24) is automatically satisfied and it is enough to take $A = 1$ in (25). To clarify Condition (25), let us consider two important special cases. When $\rho(t) = t^\alpha$ for some $0 < \alpha < 1/2$, (25) reduces to $P(\|\mu^\bullet\|_K \geq t) = o(t^{-p(\alpha)})$, with $p(\alpha) := (1/2 - \alpha)^{-1}$ and this is slightly weaker than $\mathbf{E}\|\mu^\bullet\|_K^{p(\alpha)} < \infty$. When $\rho(t) = t^{1/2} \ln^\beta(c/t)$ for some $\beta > 1/2$, then (25) is equivalent to the finiteness of $\mathbf{E} \exp(d\|\mu^\bullet\|_K^{1/\beta})$ for each $d > 0$.

Following [16], we present briefly a statistical application of Theorem 9 to the detection of epidemic change in the expectation of a random measure. In what follows, μ_k^\bullet , $k = 1, \dots, n$ are always i.i.d. copies of the *mean zero* random measure μ^\bullet . Based on the observation of the random measures $\nu_1^\bullet, \dots, \nu_n^\bullet$, we want to test the null hypothesis

$$(H_0): \nu_k^\bullet = \mu_k^\bullet, \quad k = 1, \dots, n,$$

against the so called epidemic alternative

$$(H_A) \quad \nu_k^\bullet = \begin{cases} \mu_c + \mu_k^\bullet & \text{if } k \in \mathbb{I}_n := \{k^* + 1, \dots, m^*\} \\ \mu_k^\bullet & \text{if } k \in \mathbb{I}_n^c := \{1, \dots, n\} \setminus \mathbb{I}_n \end{cases}$$

where $\mu_c \neq 0$ is some deterministic signed measure which may depend on n . To achieve this goal, we use some weighted dyadic increments statistics which behave like continuous functionals of ξ_n^\bullet in Hölder topology. Consider partial sums

$$S_n(a, b) = \sum_{na < k \leq nb} \nu_k^\bullet, \quad 0 \leq a < b \leq 1.$$

Let us denote by D_j the set of dyadic numbers in $[0, 1]$ of level j , i.e. $D_0 = \{0, 1\}$, and $D_j = \{(2l-1)2^{-j}; 1 \leq l \leq 2^{j-1}\}$, $j \geq 1$. Write for $r \in D_j$, $j \geq 0$, $r^- := r - 2^{-j}$ and $r^+ := r + 2^{-j}$. Then define the dyadic increments statistics $\text{DI}(n, \rho)$ by

$$\text{DI}(n, \rho) := \frac{1}{2} \max_{1 \leq j \leq \log_2 n} \frac{1}{\rho(2^{-j})} \max_{r \in D_j} \|S_n(r^-, r) - S_n(r, r^+)\|_K. \quad (26)$$

Here “log” stand for the logarithm with basis 2 ($\log(2^j) = j$) while “ln” denotes the natural logarithm ($\ln(e^t) = t$).

Theorem 10. *Assume that the weight function ρ satisfies (24) and that the mean zero random measure μ^\bullet satisfies (25). Then under (H_0) , $n^{-1/2}\text{DI}(n, \rho)$ converges in law to a non negative random variable Z with distribution function*

$$P(Z \leq z) = \prod_{j=1}^{\infty} \left(P(\|\gamma^\bullet\|_K \leq 2^{(j+1)/2} \rho(2^{-j}) z) \right)^{2^{j-1}}, \quad z \geq 0, \quad (27)$$

where γ^\bullet is a mean zero Gaussian random element in \mathcal{H} with the same covariance as μ^\bullet . The convergence of the product (27) is uniform on any interval $[\varepsilon, \infty)$, $\varepsilon > 0$.

Theorem 10 is easily obtained from Theorem 6 and from [16] Th. 2 and Prop. 3. For general estimates on the convergence rate in (27), see Prop. 4 in [16]. The consistency of the sequence of test statistics $n^{-1/2}\text{DI}(n, \rho)$ follows from the next result which is an easy adaptation of Th. 5 in [16].

Theorem 11. *Let ρ satisfying (24). Under (H_A) , write $l^* := m^* - k^*$ for the length of epidemics and assume that*

$$\lim_{n \rightarrow \infty} n^{1/2} \frac{u_n \|\mu_c\|_K}{\rho(u_n)} = \infty, \quad \text{where } u_n := \min\left\{\frac{l^*}{n}; 1 - \frac{l^*}{n}\right\}. \quad (28)$$

Then

$$n^{-1/2}\text{DI}(n, \rho) \xrightarrow[n \rightarrow \infty]{\text{pr}} \infty.$$

To discuss Condition (28), assume for simplicity that μ_c does not depend on n . When $\rho(t) = t^\alpha$, (28) allows us to detect *short epidemics* such that $l^* = o(n)$ and $l^* n^{-\delta} \rightarrow \infty$, where $\delta = (1 - 2\alpha)(2 - 2\alpha)^{-1}$. When $\rho(t) = t^{1/2} \ln^\beta(c/t)$ with $\beta > 1/2$, (28) is satisfied provided that $u_n = n^{-1} \ln^\gamma n$, with $\gamma > 2\beta$. This leads to detection of short epidemics such that $l^* = o(n)$ and $l^* \ln^{-\gamma} n \rightarrow \infty$. In both cases one can detect symmetrically *long epidemics* such that $n - l^* = o(n)$.

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