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Estimating the interval of epidemic change*

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Abstract

Let $X_{1,n}, X_{2,n}, \dots, X_{n,n}$ be a triangular array of independent random elements in a measurable space E. For $i \leq ns_n^*$ or $i > nt_n^*$, the $X_{i,n}$'s have distribution P_n, while for $ns_n^* < i \leq nt_n^*$, they have distribution Q_n. We estimate the vector $\theta_n = (s_n^*, t_n^*)$ of change points, by extending Dümbgen's (1991) method to this epidemic model. In the simple case where P_n and Q_n do not depend on n, our estimator $\widehat{\theta_n}$ satisfies $\widehat{\theta_n} - \theta_n = O_{\mathrm{Pr}}(n^{-1})$.

Résumé

Soit $X_{1,n}, X_{2,n}, \dots, X_{n,n}$ un tableau triangulaire d'éléments aléatoires indépendants à valeurs dans un espace mesurable E. On suppose que pour $i \leq ns_n^*$ ou $i > nt_n^*$, les $X_{i,n}$ ont pour loi P_n, alors que pour $ns_n^* < i \leq nt_n^*$, leur loi est Q_n. Nous estimons le vecteur $\theta_n = (s_n^*, t_n^*)$ des points de rupture, en généralisant la méthode de Dümbgen (1991) à ce modèle de rupture épidémique. Dans le cas simple où P_n et Q_n ne dépendent pas de n, notre estimateur $\widehat{\theta_n}$ vérifie $\widehat{\theta_n} - \theta_n = O_{\mathrm{Pr}}(n^{-1})$.

Keywords : change point location, empirical process, epidemic model.

Mathematics Subject Classifications (2000): 62G05

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1 Introduction and results

A general epidemic change point model may be described as follows. For $n = 3, 4, \ldots$, let P_n and Q_n be two probability distributions on a measurable space E and let $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ be a triangular array of independent random elements in E. For $i \le ns_n^*$ or $i > nt_n^*$, the $X_{i,n}$'s have distribution P_n, while for $ns_n^* < i \le nt_n^*$, they have distribution Q_n. Our aim is to estimate the vector $\theta_n = (s_n^*, t_n^*)$ of change points. For a discussion on the estimation of epidemic intervals, we refer to the introduction of [5] and the references therein. The unknown parameter θ_n belongs to the set

$$T_n := \bigl\{(s,t) \in \{1/n, 2/n, \ldots, n/n\}^2;\ s < t\bigr\}.$$

For notational simplification, we shall drop the index n in s_n^* and t_n^* and put

$$h^* := h_n^* = t^* - s^* = t_n^* - s_n^*. \tag{1}$$

Throughout the paper we adopt the following notations for the index sets $]k, m]$ and $]m, k]$ where k, m are integers such that $0 \le k < m \le n$.

$$\begin{aligned}]k, m] &:= \{i \in \mathbb{N};\ k < i \le m\}, && (2)\\ ]m, k] &:= \{i \in \mathbb{N};\ 0 < i \le k \text{ or } m < i \le n\}. && (3) \end{aligned}$$

Now for $(s,t) = (k/n, m/n)$, introduce the empirical measures

$$\begin{aligned} P_n^{s,t} = P_n^{k,m} &:= \frac{1}{m-k} \sum_{i \in]k,m]} \delta_{X_{i,n}}, && (4)\\ P_n^{t,s} = P_n^{m,k} &:= \frac{1}{n-(m-k)} \sum_{i \in]m,k]} \delta_{X_{i,n}}. && (5) \end{aligned}$$

We denote their pointwise expectations by

$$\Pi_n^{s,t} := \mathbf{E}\, P_n^{s,t}, \quad \Pi_n^{t,s} := \mathbf{E}\, P_n^{t,s}. \tag{6}$$

With the weight function

$$w(u) := u^{1/2}(1-u)^{1/2}, \quad u \in [0,1],$$

introduce the signed measure

$$D_n^{s,t} := w(t-s)\bigl(P_n^{s,t} - P_n^{t,s}\bigr), \quad (s,t) \in T_n \tag{7}$$

and put

$$\Delta_n^{s,t} := \mathbf{E}\, D_n^{s,t} = r_n(s,t)(Q_n - P_n), \tag{8}$$

the second equality being to be justified in the proof of Lemma 3 below. Now we choose a seminorm N_n on the space $\mathcal{M}$ of all finite signed measures on E

and note that $N_n(\Delta_n^{s,t}) = |r_n(s,t)| N_n(P_n - Q_n)$. As $|r_n|$ has a maximum on T_n reached at $(s,t) = \theta_n$, see Lemma 3, this lead us to propose the estimator

$$\widehat{\theta}_n := \arg\max \big\{ N_n(D_n^{s,t});\ (s,t) \in T_n \big\}, \tag{9}$$

which is the generalization of Dümbgen's estimator in the setting of epidemic change.

We make the same assumptions as Dümbgen [2] on P_n, Q_n and N_n, namely:

a) there is a Vapnik-Cervonenkis class $\mathcal{D}$ of measurable subsets of E such that
$$N_n(\mu) \le \|\mu\| := \sup\{|\mu(D)|;\ D \in \mathcal{D}\}, \quad n \ge 3,\ \mu \in \mathcal{M}; \tag{10}$$

b) there is a constant $C_0 > 0$ and a sequence $(\gamma_n) \in \mathbb{R}^+$ such that
$$\Pr\Big(N_n(Q_n - P_n) \ge \frac{C_0}{\gamma_n} \Big) \xrightarrow[n\to\infty]{} 1 \quad \text{and} \quad \frac{\gamma_n (\ln n)^{1/2}}{n^{1/2}} \xrightarrow[n\to\infty]{} 0. \tag{11}$$

If N_n is non random, (11) reduces to $N_n(Q_n - P_n) \ge C_0 \gamma_n^{-1}$ for n large enough. If moreover $P_n = P$ and $Q_n = Q$ do not depend on n, this in turn, reduces to $N_n(Q - P) \ge C_0$.

We assume moreover that

$$\theta_n \xrightarrow[n\to\infty]{} \theta = (s_0, t_0), \text{ with } 0 < h_0 := t_0 - s_0 < 1. \tag{12}$$

Theorem 1. *Under (12) and if the seminorms N_n and the distributions P_n and Q_n satisfy the conditions (10) and (11), then*

$$\widehat{\theta}_n - \theta_n = O_{\Pr}(\gamma_n^2 n^{-1}). \tag{13}$$

The proof of Theorem 1 reduces to the proof of the Proposition 2 whose statement requires some more notations. Equipping $\mathbb{R}^2$ with the ℓ^1 norm

$$|(s,t)|_1 := |s| + |t|,$$

we define for any positive d the ball $T_n(d)$ as

$$T_n(d) := \big\{ (s,t) \in T_n;\ |(s,t) - \theta_n|_1 \le d\gamma_n^2 n^{-1} \big\}.$$

Proposition 2. *If N_n, P_n and Q_n satisfy (10) and (11), then there is a constant $C_1 > 0$ such that*

$$\liminf_{n\to\infty} \Pr(A_n(d)) \xrightarrow[d\to\infty]{} 1, \tag{14}$$

where the event $A_n(d)$ is defined by $A_n(d) := A'_n(d) \cap A''_n$ with

$$A'_n(d) := \Big\{ N_n(D_n^{s,t}) - N_n(D_n^{\theta_n}) \le -\frac{C_1}{\gamma_n} |(s,t) - \theta_n|_1, \forall (s,t) \in T_n \setminus T_n(d) \Big\}$$

and

$$A''_n := \Big\{ N_n(D_n^{\theta_n}) \ge \frac{C_1}{\gamma_n} \Big\}.$$

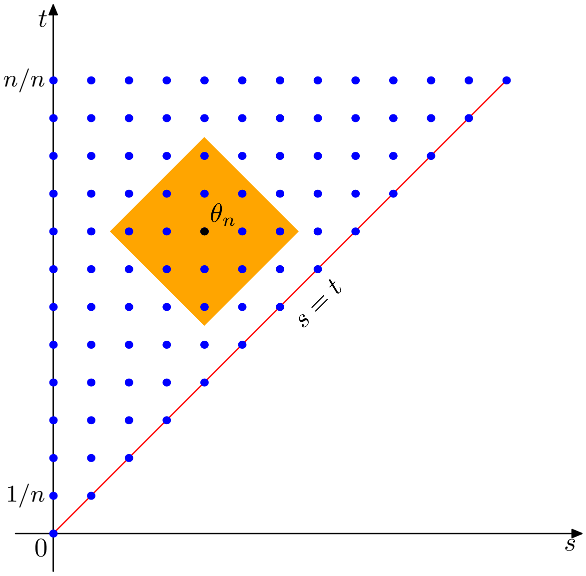

Figure 1: The sets T_n and $T_n(d)$

Reduction of Theorem 1 to Proposition 2. From (14) and the definition of $A_n(d)$, we can find for any $\varepsilon > 0$ a positive d_ε and an integer n_ε such that

$$\forall n \geq n_\varepsilon, \quad \Pr\bigl(A'_n(d_\varepsilon)\bigr) \geq 1-\varepsilon. \tag{15}$$

For every $\omega \in A'_n(d_\varepsilon)$, $(s,t) = \widehat{\theta}_n(\omega)$ satisfies at least one of the two following conditions:

$$N_n\bigl(D_n^{\widehat{\theta}_n(\omega)}\bigr) - N_n\bigl(D_n^{\theta_n}\bigr) \leq -\frac{C_1}{\gamma_n}|\widehat{\theta}_n(\omega)-\theta_n|_1, \tag{16}$$

or

$$|\widehat{\theta}_n(\omega)-\theta_n|_1 \leq d_\varepsilon \frac{\gamma_n^2}{n}. \tag{17}$$

By (9), the left hand side of (16) is non negative, hence this inequality holds true only if $|\widehat{\theta}_n(\omega)-\theta_n|_1 = 0$, in which case (17) is trivially satisfied. Consequently $\widehat{\theta}_n(\omega)$ satisfies (17) for every $\omega \in A'_n(d_\varepsilon)$ and in view of (15), this gives $\widehat{\theta}_n - \theta_n = O_{\Pr}(\gamma_n^2 n^{-1})$. $\square$

2 Proofs

Lemma 3. *The function r_n defined by (8) satisfies*

i) $|r_n|$ has a unique maximum on T_n, reached at the point $(s,t) = \theta_n$ and $|r_n(\theta_n)| = r_n(\theta_n) = w(\theta_n)$.

ii) there is a positive constant C such that for n large enough,

$$r_n(\theta_n) - r_n(s,t) \geq C|\theta_n - (s,t)|_1, \quad (s,t) \in T_n. \tag{18}$$

Proof of Lemma 3 i). Recalling the notations (4)–(8), we have for any $0 < k < m \leq n$,

$$\begin{aligned}
\mathbf{E}\, P^{k,m} - \mathbf{E}\, P^{m,k} &= \frac{1}{m-k} \sum_{i \in]k,m]} \mathbf{E}\, \delta_{X_{i,n}} - \frac{1}{n-(m-k)} \sum_{i \in]m,k]} \mathbf{E}\, \delta_{X_{i,n}} \\
&= \frac{1}{m-k} \sum_{i \in]k,m]} \big(\mathbf{1}_{]m^*,k^*]}(i) P_n + \mathbf{1}_{]k^*,m^*]}(i) Q_n \big) \\
&\quad - \frac{1}{n-(m-k)} \sum_{i \in]m,k]} \big(\mathbf{1}_{]m^*,k^*]}(i) P_n + \mathbf{1}_{]k^*,m^*]}(i) Q_n \big) \\
&= \left(\frac{\sharp\,(]k,m] \cap]m^*,k^*])}{\sharp\,]k,m]} - \frac{\sharp\,(]m,k] \cap]m^*,k^*])}{\sharp\,]m,k]} \right) P_n \\
&\quad + \left(\frac{\sharp\,(]k,m] \cap]k^*,m^*])}{\sharp\,]k,m]} - \frac{\sharp\,(]m,k] \cap]k^*,m^*])}{\sharp\,]m,k]} \right) Q_n.
\end{aligned} \tag{19}$$

It is convenient to interpret the coefficients p_n of P_n and q_n of Q_n in the following way. Denote by ν the uniform distribution on the unit circle. For any real numbers s, t such that $0 < t - s < 1$, introduce the arc of unit circle

$$\operatorname{arc}(s,t] := \big\{ \exp(2\pi i u);\ s < u \leq t \big\}$$

and denote by $\operatorname{arc}(s,t]^c$ its complementary arc, which may be represented as $\operatorname{arc}(t-1,s]$ or $\operatorname{arc}(t,s+1]$ as well. Then we have $\nu\big(\operatorname{arc}(s,t]\big) = t - s$ and $\nu\big(\operatorname{arc}(s,t]^c\big) = 1 - (t-s)$. Now the coefficient $q_n(k/n, m/n)$ of Q_n in (19) may be represented as the difference of conditional probabilities

$$q_n(s,t) = \nu\big(\operatorname{arc}(s^*,t^*] \mid \operatorname{arc}(s,t]\big) - \nu\big(\operatorname{arc}(s^*,t^*) \mid \operatorname{arc}(s,t)^c\big). \tag{20}$$

Similarly the coefficient of P_n may be written as

$$\begin{aligned}
p_n(s,t) &= \nu\big(\operatorname{arc}(s^*,t^*]^c \mid \operatorname{arc}(s,t]\big) - \nu\big(\operatorname{arc}(s^*,t^*]^c \mid \operatorname{arc}(s,t]^c\big) \\
&= \Big(1 - \nu\big(\operatorname{arc}(s^*,t^*] \mid \operatorname{arc}(s,t]\big)\Big) - \Big(1 - \nu\big(\operatorname{arc}(s^*,t^*] \mid \operatorname{arc}(s,t]^c\big)\Big) \\
&= -q_n(s,t).
\end{aligned}$$

This establishes the second equality in formula (8) with $r_n(s,t) = w(t-s) q_n(s,t)$. Moreover using the fact that

$$P(A \mid B) - P(A \mid B^c) = \frac{P(A \cap B) - P(A)P(B)}{P(B)P(B^c)}$$

for any probability measure P and any pair of events A, B, provided that $0 < P(B) < 1$, we obtain

$$r_n(s,t) = \frac{\nu\big(\mathrm{arc}(s^*,t^*]\cap \mathrm{arc}(s,t]\big) - \nu\big(\mathrm{arc}(s^*,t^*]\big)\nu\big(\mathrm{arc}(s,t]\big)}{\nu\big(\mathrm{arc}(s,t]\big)^{1/2}\nu\big(\mathrm{arc}(s,t]^c\big)^{1/2}} \tag{21}$$

$$= \frac{\nu\big(\mathrm{arc}(s^*,t^*]\cap \mathrm{arc}(s,t]\big) - (t^*-s^*)(t-s)}{(t-s)^{1/2}(1-(t-s))^{1/2}}, \tag{22}$$

for $0 < t-s < 1$. As a function defined on the open band $0 < t-s < 1$ of the plane with coordinates (s,t), r_n is $(1,1)$-periodic because the transformation $s \mapsto s+1$ corresponds to the identity on the unit circle, so

$$r_n(s+1,t+1) = r_n(s,t). \tag{23}$$

Next we observe that by $(s,t) \mapsto (t-1,s)$, $\mathrm{arc}(s,t]$ is transformed into its complementary arc, from which it easily follows that

$$r_n(t-1,s) = -r_n(s,t). \tag{24}$$

Another useful transform of the unit circle is the symmetry $s \mapsto s' = s^*+t^*-s$ with respect to the "middle" $(s^*+t^*)/2$ of $\mathrm{arc}(s^*,t^*]$. This transform preserves the measure ν, leave $\mathrm{arc}(s^*,t^*]$ inchanged up to its null-measure endpoints and change $\mathrm{arc}(s,t]$ into $\mathrm{arc}[t',s')$. From this we easily deduce that

$$r_n(s^*+t^*-t, s^*+t^*-s) = r_n(s,t). \tag{25}$$

In terms of coordinates transform, the map $(s,t) \mapsto (t',s')$ is the symmetry with axis the straight line D^* with equation $s+t = s^*+t^*$. In particular the level lines $r_n = \lambda$ are symmetric with respect to D^*, see figure 3.

Combining (24) and (25) leads to

$$r_n(s^*+t^*-s-1, s^*+t^*-t) = -r_n(s,t). \tag{26}$$

The corresponding transform of the couple (s,t) is the symmetry with respect to the point $I = \big(\frac{s^*+t^*-1}{2}, \frac{s^*+t^*}{2}\big)$. In particular the level lines $r_n = \lambda$ and $r_n = -\lambda$ are symmetric with respect to I.

The level lines $r_n = 0$ are easily seen to be the segments

$$(\varsigma_1)\left\{\begin{array}{l}(1-h^*)t+h^*s = s^*\\ 0<t-s<1\end{array}\right. \qquad (\varsigma_2)\left\{\begin{array}{l}h^*t+(1-h^*)s = t^*\\ 0<t-s<1\end{array}\right.$$

and all their images by the translations by vectors (j,j), $j \in \mathbb{Z}$. The open segments ς_1 and ς_2 have endpoints (t^*-1,t^*) and (s^*,s^*) for the first one, (t^*-1,t^*) and (s^*,s^*+1) for the second one. Note also that r_n could be extended by continuity to the closed band $0 \le t-s \le 1$, putting $r_n(s,t) := 0$, along the straight lines $t-s = 0$ and $t-s = 1$.

Denote by $\mathrm{Tpz}(0)$ the open trapezium with vertices these four endpoints (so ς_1 and ς_2 are opposite non parallel sides of $\mathrm{Tpz}(0)$). Similarly define $\mathrm{Tpz}(-1)$

as the symmetric of Tpz(0) with respect to I. For $i = 2j$, $j \in \mathbb{Z} \setminus \{0\}$, define Tpz($i$) as Tpz(0) shifted by the vector (j, j) and for $i = 2j + 1$, define Tpz(i) as Tpz(-1) shifted by the vector $(j + 1, j + 1)$. We note that (s^*, t^*) belongs to Tpz(0) and that $r_n(s^*, t^*) = w(h^*)$ is positive. By an obvious connexity argument, it follows that r_n is positive on each Tpz(i) for i even and negative on each Tpz(i) for i odd, as represented in figure 2.

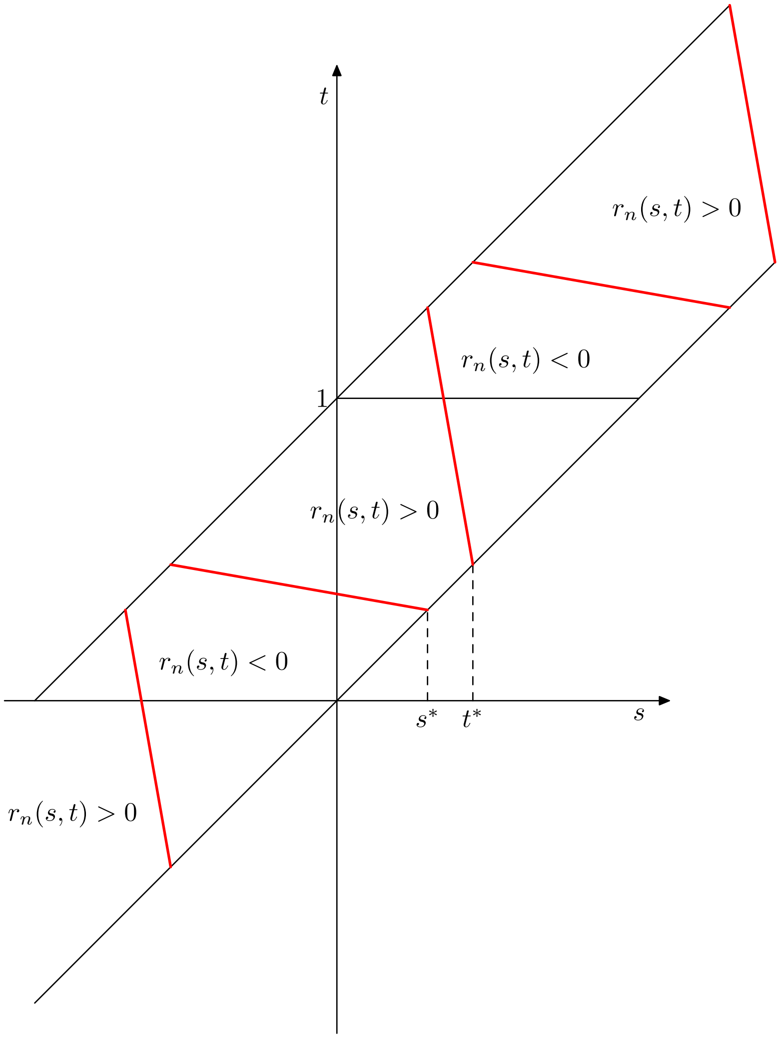

Figure 2: Sign of $r_n(s, t)$

Now let us look at r_n restricted to Tpz(0). The level lines $r_n = \lambda$ on Tpz(0) are represented by figure 3. To describe them more precisely, we introduce the following notations. Denote by “West”, “South”, “East” and “North” the segments joining (s^*, t^*) to respectively (t^*-1, t^*), (s^*, t^*), (s^*, t^*), (s^*, s^*+1). Define "South-West" as the subdomain of Tpz(0) with edges South and West and define similarly "South-East", "North-East" and "North-West".

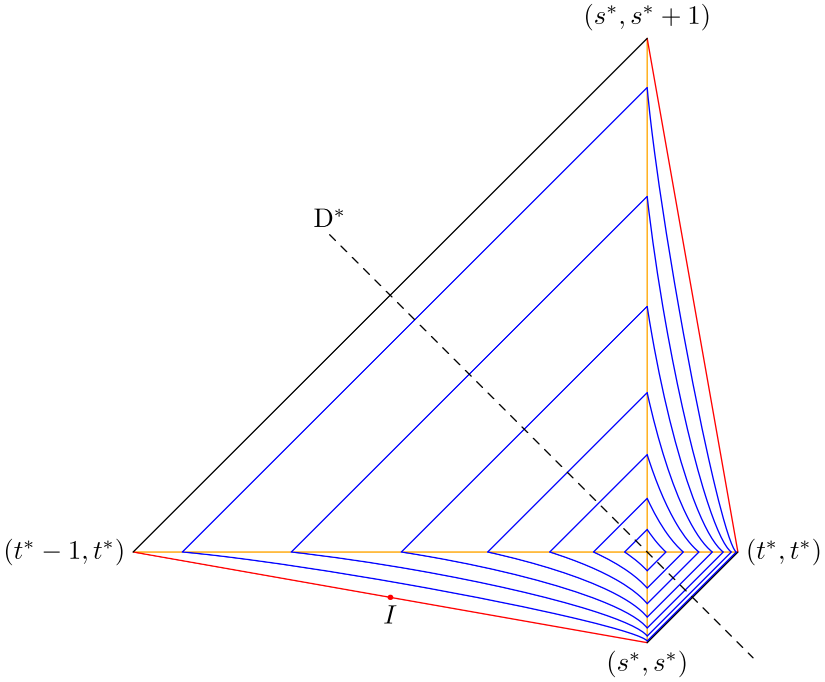

Figure 3: Level lines (in blue) of r_n in Tpz(0)

$\square$

In the subdomain "South-East" of Tpz(0), i.e. when $s^* \le s < t \le t^*$, r_n writes (recalling that $h^* = t^* - s^*$)

$$r_n(s,t) = \frac{(t-s) - (t-s)h^*}{(t-s)^{1/2}(1-(t-s))^{1/2}} = (1-h^*)\Big(\frac{t-s}{1-t+s}\Big)^{1/2}. \qquad (27)$$

It follows that the level lines $r_n(s,t) = \lambda$ for $\lambda > 0$ in "South-East" part of Tpz(0) are the segments of straight line

$$L_{\mathrm{SE}}(\lambda): \quad t - s = \frac{\lambda^2}{\lambda^2 + (1-h^*)^2}, \quad s^* \le s < t \le t^*.$$

As $\lambda^2(\lambda^2 + (1-h^*)^2)^{-1}$ increases in λ, the maximal value for λ is obtained when $t - s$ is maximal in this subdomain, i.e. when $\lambda^2(\lambda^2 + (1-h^*)^2)^{-1} = h^*$. Hence the maximal level line in "South-East" is obtained for $\lambda = w(h^*)$. As this line reduce to the point (s^*, t^*), r_n has a unique maximum in "South-East" part of Tpz(0), located at this point.

In the “South-West” part of Tpz(0), i.e. when $t^* - 1 < s \leq s^*$, $t \leq t^*$ and $(1-h^*)t + h^*s \geq s^*$, r_n writes

$$r_n(s,t) = \frac{(t-s^*) - (t-s)h^*}{(t-s)^{1/2}(1-(t-s))^{1/2}} = \frac{(1-h^*)t + h^*s - s^*}{(t-s)^{1/2}(1-(t-s))^{1/2}}.$$

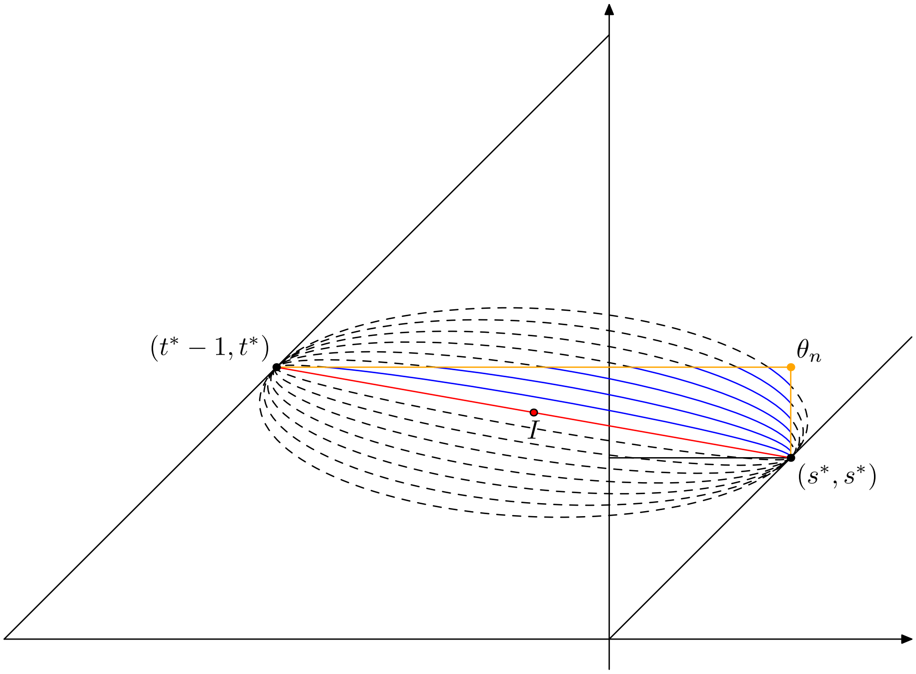

Figure 4: Level lines of r_n in “South-West” part of Tpz(0)

The level line $L_{\mathrm{SW}}(\lambda)$: $r_n(s,t) = \lambda$ for $\lambda > 0$ in this part of Tpz(0) is the arc of the ellipse

$$E(\lambda): \quad \big((1-h^*)t + h^*s - s^*\big)^2 = \lambda^2(t-s)(1-(t-s))$$

obtained as the intersection of $E(\lambda)$ with “South-West” part of Tpz(0). Each $E(\lambda)$ pass through the points (s^*, t^*) and (t^*-1, t^*) where they are respectively tangential to the straight lines $t - s = 0$ and $t - s = 1$.

In addition to (s^*, t^*), the ellipse $E(\lambda)$ cross the South edge $s = s^*$ at the point (s^*, t_λ), with

$$t_\lambda = s^* + \frac{\lambda^2}{\lambda^2 + (1-h^*)^2}.$$

This is also the intersection point of $L_{\mathrm{SE}}(\lambda)$ with South edge. Clearly the maximal value of λ for wich such an intersection exists is given for $t_\lambda = t^*$ which leads again to $\lambda = w(h^*)$. To confirm this result, let us note that the second intersection point of $E(\lambda)$ with the East edge $t = t^*$ is the point (s_λ, t^*), where

$$s_\lambda = t^* - \frac{{h^*}^2}{\lambda^2 + {h^*}^2}.$$

Noting that s_λ is an increasing function of λ and that it is maximal for $s_\lambda = s^*$, we obtain again $w(h^*)$ as the maximal value for λ. For this value of λ the arc of ellipse $L_{\mathrm{SE}}(\lambda)$ reduces to the single point (s^*, t^*), so r_n has a unique maximum in the "South-West" part of Tpz(0), reached at (s^*, t^*).

Using the symmetry of the level lines of r_n with respect to D*, see (25), we conclude that this result extends to the whole domain Tpz(0). Using the symmetry of level lines $r_n = \lambda$ and $r_n = -\lambda$ with respect to the point $I = \big(\frac{s^*+t^*-1}{2}, \frac{s^*+t^*}{2}\big)$, see (26), we deduce from this that r_n which is everywhere negative on Tpz(-1), takes a minimal value $-w(h^*)$ on Tpz(-1), located at the unique point $(t^* - 1, s^*)$. Due to the $(1, 1)$ periodicity of r_n, the same holds true for Tpz(1) with location of the minimum at $(t^*, s^* + 1)$. Finally, as neither $(t^* - 1, s^*)$ nor $(t^*, s^* + 1)$ belongs to the triangle $0 \leq s \leq t \leq 1$, we conclude that $|r_n|$ has a unique maximum in T_n, located at $\theta_n = (s^*, t^*)$.

Proof of Lemma 3 ii). We have to check that

$$f_n(s, t) := \frac{r_n(\theta_n) - r_n(s, t)}{|\theta_n - (s, t)|_1}$$

has a positive lower bound on $T_n \setminus \{\theta_n\}$.

First we look for such a bound on Tpz(0) and we start with its South-East part. Noting that in this subdomain, $|\theta_n - (s, t)|_1 = h^* - (t - s)$ and using (27), we have

$$\begin{aligned} f_n(s, t) &= \frac{(1 - h^*)^{1/2}\big({h^*}^{1/2}(1 - (t - s))^{1/2} - (1 - h^*)^{1/2}(t - s)^{1/2}\big)}{(h^* - (t - s))(1 - (t - s))^{1/2}} \\ &= \frac{(1 - h^*)^{1/2}\big(h^*(1 - (t - s)) - (1 - h^*)(t - s)\big)}{(h^* - (t - s))(1 - (t - s))^{1/2}\big({h^*}^{1/2}(1 - (t - s))^{1/2} + (1 - h^*)^{1/2}(t - s)^{1/2}\big)} \\ &= \frac{(1 - h^*)^{1/2}}{(1 - (t - s))^{1/2}\big({h^*}^{1/2}(1 - (t - s))^{1/2} + (1 - h^*)^{1/2}(t - s)^{1/2}\big)} \\ &= \frac{(1 - h^*)^{1/2}}{{h^*}^{1/2}(1 - (t - s)) + (1 - h^*)^{1/2}w(t - s)}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality to the denominator in this last expression, we see that it is less than

$$\sqrt{h^* + 1 - h^*}\sqrt{(1 - (t - s))^2 + (t - s)(1 - (t - s))} = \sqrt{1 - (t - s)} \leq 1.$$

This leads to

$$f_n(s, t) \geq (1 - h^*)^{1/2} \quad \text{on South-East part of Tpz(0).} \tag{28}$$

Symmetrically we have on the North-West part of Tpz(0),

$$f_n(s,t) = \frac{{h^*}^{1/2}}{(1-h^*)^{1/2}(t-s) + {h^*}^{1/2} w(t-s)},$$

so the same trick leads to the lower bound $f_n(s,t) \geq {h^*}^{1/2}/\sqrt{t-s}$, whence

$$f_n(s,t) \geq {h^*}^{1/2} \quad \text{on North-West part of Tpz(0).} \tag{29}$$

To control $f_n(s,t)$ on the South-West part of Tpz(0), we simply note that along the level line $L_{SW}(\lambda)$ we have

$$f_n(s,t) = \frac{w(h^*) - \lambda}{s^* + t^* - (s+t)}.$$

This ratio is minimal on $L_{SW}(\lambda)$ when $s^* + t^* - (s+t)$ is maximal that is when (s,t) is one of the endpoints of the arc of ellipse $L_{SW}(\lambda)$. More precisely this maximum is reached when $(s,t) = (s_\lambda, t^*)$ is on the West edge if $h^* < 1/2$ and when $(s,t) = (s^*, t_\lambda)$ is on the South edge if $h^* \geq 1/2$. Both cases are already covered by the lower bounds obtained for South-East and North-West parts of Tpz(0). Of course the same holds true for the North-East part of Tpz(0) due to the symmetry of the level lines of r_n and of the function $|\theta_n - (s,t)|_1$ with respect to D*. All this leads to

$$f_n(s,t) \geq \min\left({h^*}^{1/2}, (1-h^*)^{1/2}\right) \geq {h^*}^{1/2}(1-h^*)^{1/2} \quad \text{on Tpz(0).} \tag{30}$$

In $T_n \setminus \mathrm{Tpz}(0)$, r_n is negative, so $r_n(\theta_n) - r_n(s,t) \geq r_n(\theta_n)$. Noting also that for any $(s,t) \in T_n$, $|\theta_n - (s,t)|_1$ is at most 2, we obtain $f_n(s,t) \leq w(h^*)/2$ on this subdomain. Accounting (30), we conclude that

$$\forall (s,t) \in T_n \setminus \{\theta_n\}, \quad \frac{r_n(\theta_n) - r_n(s,t)}{|\theta_n - (s,t)|_1} \geq \frac{1}{2} w(h^*). \tag{31}$$

Using (12) and the continuity of w, (18) follows for large enough n. □

Proof of Proposition 2. We shall split the proof in 5 steps. Write

$$\widetilde{w}\big((s,t)\big) := w(t-s), \text{ whence } \widetilde{w}(\theta_n) = (t^* - s^*)^{1/2}\big(1 - (t^* - s^*)\big)^{1/2}. \tag{32}$$

Set

$$B_n^{s,t} := D_n^{s,t} - \Delta_n^{s,t}, \tag{33}$$

with $D_n^{s,t}$ and $\Delta_n^{s,t}$ defined by (7) and (8) respectively.

Step 1. We first check that

$$\left| N_n(D_n^{\theta_n}) - \widetilde{w}(\theta_n) N_n(Q_n - P_n) \right| \leq \|B_n^{\theta_n}\| = O_{\mathrm{Pr}}(n^{-1/2}). \tag{34}$$

Recalling (7), we have $N_n(\Delta_n^{\theta_n}) = \widetilde{w}(\theta_n) N_n(Q_n - P_n)$. As N_n is a seminorm, we get

$$\big| N_n(D_n^{\theta_n}) - N_n(\Delta_n^{\theta_n}) \big| \le N_n(D_n^{\theta_n} - \Delta_n^{\theta_n}) = N_n(B_n^{\theta_n}) \le \| B_n^{\theta_n} \|.$$

Hence to obtain (34), it suffices to prove that $\|B_n^{\theta_n}\| = O_{\mathrm{Pr}}(n^{-1/2})$.

Introduce now for $0 \le s < t \le 1$, the centered random measures

$$S_n(s,t) := \sum_{ns < i \le nt} \big(\delta_{X_{i,n}} - \mathcal{L}(X_{i,n}) \big), \qquad S_n(t,s) := -S_n(s,t), \tag{35}$$

$$S_n'(s,t) := \sum_{\substack{0 < i \le ns \\ \text{or } nt < i \le n}} \big(\delta_{X_{i,n}} - \mathcal{L}(X_{i,n}) \big), \qquad S_n'(t,s) := -S_n'(s,t), \tag{36}$$

where $\mathcal{L}(X_{i,n})$ denotes the distribution of $X_{i,n}$ which equals either P_n or Q_n. With these notations we have

$$B_n^{s,t} = w(t-s) \left(\frac{S_n(s,t)}{n(t-s)} - \frac{S_n'(s,t)}{n(1-(t-s))} \right)$$

whence

$$B_n^{\theta_n} = \left(\frac{1-(t^*-s^*)}{t^*-s^*} \right)^{1/2} \frac{S_n(s^*,t^*)}{n} - \left(\frac{t^*-s^*}{1-(t^*-s^*)} \right)^{1/2} \frac{S_n'(s^*,t^*)}{n}. \tag{37}$$

By the maximal inequality (7.4) in Lemma 2 of [2], we have

$$\mathrm{Pr}\left(\frac{\|S_n(s^*,t^*)\|}{n} \ge \eta \frac{(t^*-s^*)^{1/2}}{n^{1/2}} \right) \le 2K_1 \exp\Big(-K_2 \eta^2/4 \Big).$$

Choosing $\eta = c\big(1-(t^*-s^*)\big)^{-1/2}$ and noting that $0 < 1-(t^*-s^*) < 1$, we obtain

$$\begin{aligned} \mathrm{Pr}\left(\left(\frac{1-(t^*-s^*)}{t^*-s^*} \right)^{1/2} \frac{\|S_n(s^*,t^*)\|}{n} \ge \frac{c}{n^{1/2}} \right) &\le 2K_1 \exp\left(-\frac{K_2 c^2}{4(1-(t^*-s^*))} \right) \\ &\le 2K_1 \exp\left(-\frac{K_2}{4} c^2 \right). \end{aligned}$$

For every $\varepsilon > 0$, there is a $c = c(\varepsilon)$ such that $2K_1 \exp(-K_2 c^2/4) < \varepsilon$, whence

$$\left(\frac{1-(t^*-s^*)}{t^*-s^*} \right)^{1/2} \frac{S_n(s^*,t^*)}{n} = O_{\mathrm{Pr}}(n^{-1/2}).$$

It is clear that the same estimate holds true for the second term in the right hand side of (37) because in the maximal inequality above the order structures on the indexes i plays no rôle, only the number of terms matters. Thus the verification of (34) is complete. □

Step 2. We show now that for some constant $C' > 0$,

$$\Pr\Big(N_n(D_n^{s,t})-N_n(D_n^{\theta_n}) \le \|B_n^{s,t}-B_n^{\theta_n}\|-\frac{C'}{\gamma_n}|(s,t)-\theta_n|_1,\ \forall (s,t)\in T_n\Big) \xrightarrow[n\to\infty]{} 1. \tag{38}$$

Recalling (32) and (12), we note that $\widetilde{w}(\theta_n)$ converges to $\widetilde{w}(\theta)$ which belongs to $(0,1)$. Combining this with (11), we obtain for any constant C_1 in $(0, C_0\widetilde{w}(\theta))$,

$$\Pr\Big(\widetilde{w}(\theta_n)N_n(Q_n-P_n) \ge \frac{C_1}{\gamma_n}\Big) \xrightarrow[n\to\infty]{} 1. \tag{39}$$

From (34), (39) and the fact that by (11), $n^{-1/2} = o(\gamma_n^{-1})$, we deduce that

$$\Pr\big(N_n(D_n^{\theta_n}) \ge C_1\gamma_n^{-1}\big) \xrightarrow[n\to\infty]{} 1, \quad 0 < C_1 < C_0\widetilde{w}(\theta). \tag{40}$$

From (8) and the fact thact that $r_n(\theta_n) = \widetilde{w}(\theta_n)$ we get $Q_n - P_n = \Delta_n^{\theta_n}/\widetilde{w}(\theta_n)$, whence

$$\Delta_n^{s,t} = \frac{r_n(s,t)}{\widetilde{w}(\theta_n)}\Delta_n^{\theta_n}.$$

Recalling that $D_n^{s,t} = B_n^{s,t} + \Delta_n^{s,t}$, we can write

$$\begin{aligned}
D_n^{s,t} &= B_n^{s,t} - B_n^{\theta_n} + B_n^{\theta_n} + \frac{r_n(s,t)}{\widetilde{w}(\theta_n)}\Delta_n^{\theta_n} \\
&= B_n^{s,t} - B_n^{\theta_n} - \frac{r_n(\theta_n)-r_n(s,t)}{\widetilde{w}(\theta_n)}\Delta_n^{\theta_n} + \frac{r_n(\theta_n)}{\widetilde{w}(\theta_n)}\Delta_n^{\theta_n} + B_n^{\theta_n} \\
&= B_n^{s,t} - B_n^{\theta_n} - \big(r_n(\theta_n)-r_n(s,t)\big)\frac{B_n^{\theta_n}}{\widetilde{w}(\theta_n)} - \frac{r_n(\theta_n)-r_n(s,t)}{\widetilde{w}(\theta_n)}D_n^{\theta_n} \\
&\qquad + \frac{r_n(\theta_n)}{\widetilde{w}(\theta_n)}\Delta_n^{\theta_n} + B_n^{\theta_n} \\
&= B_n^{s,t} - B_n^{\theta_n} - \big(r_n(\theta_n)-r_n(s,t)\big)\frac{B_n^{\theta_n}}{\widetilde{w}(\theta_n)} - \frac{r_n(\theta_n)}{\widetilde{w}(\theta_n)}B_n^{\theta_n} \\
&\qquad + \frac{r_n(s,t)}{\widetilde{w}(\theta_n)}D_n^{\theta_n} + B_n^{\theta_n}.
\end{aligned}$$

Finally

$$D_n^{s,t} = B_n^{s,t} - B_n^{\theta_n} - \big(r_n(\theta_n)-r_n(s,t)\big)\frac{B_n^{\theta_n}}{\widetilde{w}(\theta_n)} + \frac{r_n(s,t)}{\widetilde{w}(\theta_n)}D_n^{\theta_n}.$$

By triangular inequality for the seminorm N_n and domination of N_n by the norm $\|\ \|$, it follows

$$N_n(D_n^{s,t}) \le \|B_n^{s,t} - B_n^{\theta_n}\| + \big(r_n(\theta_n)-r_n(s,t)\big)\frac{\|B_n^{\theta_n}\|}{\widetilde{w}(\theta_n)} + \frac{r_n(s,t)}{\widetilde{w}(\theta_n)}N_n(D_n^{\theta_n}).$$

Substracting the equality $N_n(D_n^{\theta_n}) = \frac{r_n(\theta_n)}{\widetilde{w}(\theta_n)}N_n(D_n^{\theta_n})$, this gives

$$N_n(D_n^{s,t}) - N_n(D_n^{\theta_n}) \le \|B_n^{s,t} - B_n^{\theta_n}\| + \frac{r_n(\theta_n)-r_n(s,t)}{\widetilde{w}(\theta_n)}\big(\|B_n^{\theta_n}\| - N_n(D_n^{\theta_n})\big).$$

Now from (34) and (11) we have

$$N_n(D_n^{\theta_n}) \geq \widetilde{w}(\theta_n) N_n(Q_n - P_n) - \|B_n^{\theta_n}\| \geq C_0 \frac{\widetilde{w}(\theta_n)}{\gamma_n} - \|B_n^{\theta_n}\|,$$

with probability tending to 1 as n goes to infinity. Plugging this lower bound in probability in the above estimate of $N_n(D_n^{s,t}) - N_n(D_n^{\theta_n})$, we obtain the following inequalities, which are to be understood "with probability tending to 1":

$$\begin{aligned} N_n(D_n^{s,t}) - N_n(D_n^{\theta_n}) &\leq \|B_n^{s,t} - B_n^{\theta_n}\| + \frac{r_n(\theta_n) - r_n(s,t)}{\widetilde{w}(\theta_n)} \Big(2\|B_n^{\theta_n}\| - C_0 \frac{\widetilde{w}(\theta_n)}{\gamma_n} \Big) \\ &\leq \|B_n^{s,t} - B_n^{\theta_n}\| - \frac{r_n(\theta_n) - r_n(s,t)}{\gamma_n} \Big(C_0 - \frac{2\gamma_n \|B_n^{\theta_n}\|}{\widetilde{w}(\theta_n)} \Big) \\ &\leq \|B_n^{s,t} - B_n^{\theta_n}\| - \frac{r_n(\theta_n) - r_n(s,t)}{\gamma_n} \big(C_0 - o_{\mathrm{Pr}}(1) \big), \end{aligned}$$

since by (34), $\|B_n^{\theta_n}\| = O_{\mathrm{Pr}}(n^{-1/2})$ and by (11), $n^{-1/2}\gamma_n = o(1)$. Combining this upper bound with (18), we obtain (38). □

Step 3. In this step we explain why the proof of Proposition 2 reduces now to the proof of

$$\lim_{d\to\infty} \liminf_{n\to\infty} \Pr\Big(\|B_n^{s,t} - B_n^{\theta_n}\| \leq C''' \frac{|(s,t) - \theta_n|_1}{\gamma_n},\ \forall (s,t) \in T_{n,a} \setminus T_n(d) \Big) = 1, \tag{41}$$

for any constant $C''' > 0$, where $T_{n,a} := \{(s,t) \in T_n;\ a \leq t - s \leq 1 - a\}$ and a denotes any positive constant such that $a < t_0 - s_0$, see (12).

Using exponential inequality to control $S_n(s,t)$, it is not difficult to see that $\|B_n^{s,t} - B_n^{\theta_n}\|$ is bounded in probability uniformly on T_n by $n^{-1/2}L(n)$. Hence when $|(s,t) - \theta_n|_1$ is bounded from below by a positive constant, $\|B_n^{s,t} - B_n^{\theta_n}\|$ is negligible compared to $C'\gamma_n^{-1}|(s,t) - \theta_n|_1$. This happens in particular when (s,t) satisfies either $0 < t - s \leq a$ or $t - s \geq 1 - a$. Then we can find a constant $C'' \in (0, C')$ such that

$$\Pr\Big(N_n(D_n^{s,t}) - N_n(D_n^{\theta_n}) \leq -\frac{C''}{\gamma_n}|(s,t) - \theta_n|_1,\ \forall (s,t) \in T_n \setminus T_{n,a} \Big) \xrightarrow[n\to\infty]{} 1. \tag{42}$$

Next for $(s,t) \in T_{n,a}$, (s,t) may be close to θ_n, so we shall no more neglect $\|B_n^{s,t} - B_n^{\theta_n}\|$ compared to $C'\gamma_n^{-1}|(s,t) - \theta_n|_1$. Obviously (38) implies that

$$\Pr\Big(N_n(D_n^{s,t}) - N_n(D_n^{\theta_n}) \leq \|B_n^{s,t} - B_n^{\theta_n}\| - \frac{C'}{\gamma_n}|(s,t) - \theta_n|_1,\ \forall (s,t) \in T_{n,a} \setminus T_n(d) \Big)$$

tends to 1 as n tends to infinity. Then we have

$$\begin{aligned} &\Pr\Big(N_n(D_n^{s,t}) - N_n(D_n^{\theta_n}) \leq -\frac{C''}{\gamma_n}|(s,t) - \theta_n|_1,\ \forall (s,t) \in T_{n,a} \setminus T_n(d) \Big) \geq \\ &\Pr\Big(\|B_n^{s,t} - B_n^{\theta_n}\| - \frac{C'}{\gamma_n}|(s,t) - \theta_n|_1 \leq -\frac{C''}{\gamma_n}|(s,t) - \theta_n|_1,\ \forall (s,t) \in T_{n,a} \setminus T_n(d) \Big) - o(1) \\ &= \Pr\Big(\|B_n^{s,t} - B_n^{\theta_n}\| \leq \frac{C' - C''}{\gamma_n}|(s,t) - \theta_n|_1,\ \forall (s,t) \in T_{n,a} \setminus T_n(d) \Big) - o(1). \end{aligned}$$

Hence with $C''' := C' - C'' > 0$ and accounting (42), the proof of Proposition 2 is reduced to the proof of (41). ☐

Step 4. Next we reduce the proof of (41) to that of

$$\lim_{d\to\infty} \liminf_{n\to\infty} \Pr\Big(\forall (s,t) \in T_{n,a} \setminus T_n(d),$$
$$\|S_n(s,s^*) + S_n(t^*,t)\| \le K \frac{n}{\gamma_n} |(s,t) - \theta_n|_1\Big) = 1, \qquad (43)$$

for any positive constant K, where S_n denote the sums of centered random measures defined by (35).

First we need to express the increment $B_n^{s,t} - B_n^{\theta_n}$ in terms of S_n, S'_n. It is convenient to write

$$B_n^{s,t} = A_n(s,t) - A'_n(s,t),$$

with

$$A_n := v\frac{S_n}{n}, \quad A'_n := \frac{1}{v}\frac{S'_n}{n}, \quad v(s,t) := \Big(\frac{1-(t-s)}{t-s}\Big)^{1/2}.$$

Then we have

$$\begin{aligned} A_n(s,t) - A_n(\theta_n) &= \big(v(s,t) - v(\theta_n)\big)\frac{S_n(s,t)}{n} + v(\theta_n)\frac{S_n(s,t) - S_n(\theta_n)}{n} \\ &= \big(v(s,t) - v(\theta_n)\big)\frac{S_n(s,t)}{n} + v(\theta_n)\frac{S_n(s,s^*) + S_n(t^*,t))}{n} \end{aligned}$$

and similarly

$$A'_n(s,t) - A'_n(\theta_n) = \Big(\frac{1}{v(s,t)} - \frac{1}{v(\theta_n)}\Big)\frac{S'_n(s,t)}{n} + \frac{1}{v(\theta_n)}\frac{S'_n(s,s^*) + S'_n(t^*,t))}{n}.$$

This leads to

$$B_n^{s,t} - B_n^{\theta_n} = \big(v(s,t) - v(\theta_n)\big)\frac{S_n(s,t)}{n} - \Big(\frac{1}{v(s,t)} - \frac{1}{v(\theta_n)}\Big)\frac{S'_n(s,t)}{n} + R_n(s,t)$$

where

$$R_n(s,t) := v(\theta_n)\frac{S_n(s,s^*) + S_n(t^*,t)}{n} - \frac{1}{v(\theta_n)}\frac{S'_n(s,s^*) + S'_n(t^*,t)}{n}.$$

Let us note that this expression of R_n can be simplified by using the equality $S'_n(s,t) = S_n(0,1) - S_n(s,t)$, which gives

$$\begin{aligned} S'_n(s,s^*) + S'_n(t^*,t) &= S_n(0,1) - S_n(s,s^*) - S'_n(t,t^*) \\ &= S_n(0,1) - S_n(s,s^*) - S_n(0,1) + S_n(t,t^*) \\ &= -S_n(s,s^*) - S_n(t^*,t), \end{aligned}$$

whence

$$\begin{aligned} R_n(s,t) &= \Big(v(\theta_n) + \frac{1}{v(\theta_n)}\Big)\frac{S_n(s,s^*) + S_n(t^*,t)}{n} \\ &= \frac{1}{\widetilde{w}(\theta_n)}\frac{S_n(s,s^*) + S_n(t^*,t)}{n}. \end{aligned} \qquad (44)$$

The functions v and $1/v$ are uniformly Lipschitz continuous on $T_{n,a}$. Hence we can find a constant κ_a depending only on a such that for every $n \geq 2$ and every $(s,t) \in T_{n,a}$,

$$\|B_n^{s,t} - B_n^{\theta_n} - R_n(s,t)\| \leq \kappa_a |(s,t) - \theta_n|_1 \max_{x \in [0,1]} \frac{\|S_n(x, s^*)\|}{n}.$$

By the maximal inequality (7.4) in Lemma 2 of [2], it follows that for every (s,t) in $T_{n,a}$,

$$\|B_n^{s,t} - B_n^{\theta_n} - R_n(s,t)\| \leq \kappa_a Z_n |(s,t) - \theta_n|_1, \text{ with } Z_n = O_{\mathrm{Pr}}(n^{-1/2}), \tag{45}$$

where the random variable Z_n do not depend on (s,t).

From (45), we deduce that for any positive constant C''',

$$\begin{aligned}
&\Pr\Big(\|B_n^{s,t} - B_n^{\theta_n}\| \leq \frac{C'''}{\gamma_n}|(s,t) - \theta_n|_1, \ \forall (s,t) \in T_{n,a} \setminus T_n(d)\Big) \geq \\
&\Pr\Big(\|R_n(s,t)\| \leq \Big(\frac{C'''}{\gamma_n} - \kappa_a Z_n\Big)|(s,t) - \theta_n|_1, \ \forall (s,t) \in T_{n,a} \setminus T_n(d)\Big) = \\
&\Pr\Big(\|S_n(s,s^*) + S_n(t^*,t)\| \leq n\widetilde{w}(\theta_n)\Big(\frac{C'''}{\gamma_n} - \kappa_a Z_n\Big)|(s,t) - \theta_n|_1, \\
&\qquad \forall (s,t) \in T_{n,a} \setminus T_n(d)\Big).
\end{aligned}$$

Now, put $\varepsilon_n := \Pr(2\kappa_a \gamma_n Z_n \geq C''')$. Reminding that θ_n converges to $\theta = (s_0, t_0)$ with $0 < t_0 - s_0 < 1$, we have some integer n_0 such that for $n \geq n_0$, $\widetilde{w}(\theta_n) \geq \widetilde{w}(\theta)/2$. Then we obtain

$$\begin{aligned}
&\Pr\Big(\|B_n^{s,t} - B_n^{\theta_n}\| \leq \frac{C'''}{\gamma_n}|(s,t) - \theta_n|_1, \ \forall (s,t) \in T_{n,a} \setminus T_n(d)\Big) \geq \\
&\Pr\Big(\forall (s,t) \in T_{n,a} \setminus T_n(d), \ \|S_n(s,s^*) + S_n(t^*,t)\| \leq \frac{C'''\widetilde{w}(\theta)n}{4\gamma_n}|(s,t) - \theta_n|_1\Big) - \varepsilon_n.
\end{aligned}$$

In view of (11) and of the estimate $Z_n = O_{\mathrm{Pr}}(n^{-1/2})$, ε_n converges to zero. Now, with $K = C'''\widetilde{w}(\theta)/4$, it is clear that (43) implies (41). □

Step 5. Switching to the complementary event in (43), it remains to prove that

$$\begin{aligned}
\lim_{d \to \infty} \limsup_{n \to \infty} \Pr\Big(&\exists (s,t) \in T_{n,a} \setminus T_n(d), \\
&\|S_n(s,s^*) + S_n(t^*,t)\| > K\frac{n}{\gamma_n}|(s,t) - \theta_n|_1\Big) = 0,
\end{aligned} \tag{46}$$

for any positive constant K. So, let us look for a suitable upper bound for

$$p := \Pr\Big(\exists (s,t) \in T_{n,a} \setminus T_n(d), \ \|S_n(s,s^*) + S_n(t^*,t)\| > \frac{Kn}{\gamma_n}|(s,t) - \theta_n|_1\Big). \tag{47}$$

Following the notational simplifications and the argument given in [2], let us introduce for fixed $n \geq 2$, the sequences of independent random variables $(Y_i)_{i \geq 1}$

and $(Y_i^*)_{i\geq 1}$, where the Y_i's have distribution P_n and the Y_i^*'s have distribution Q_n. Define then

$$Z_m := \sum_{i=1}^{m} \big(\delta_{Y_i} - P_n\big), \quad Z_m^* := \sum_{i=1}^{m} \big(\delta_{Y_i^*} - Q_n\big).$$

By Lemma 4 below, the convergence (46) will be established if we prove that for arbitrary constant $K' > 0$,

$$\lim_{d'\to\infty} \limsup_{n\to\infty} \Pr\Big(\max_{m\geq d'\gamma_n^2} m^{-1}\|Z_m\| \geq \frac{K'}{\gamma_n}\Big) = 0, \tag{48}$$

together with the same convergence with Z_m^* replacing Z_m. Now the proof is finished exactly as in [2]. We remind the argument for reading convenience. The sequences $(m^{-1}\|Z_m\|)_{m\geq 1}$ and $(m^{-1}\|Z_m^*\|)_{m\geq 1}$ are reverse submartingales. Then by Chow's inequality we have

$$\Pr\Big(\max_{m\geq d'\gamma_n^2} m^{-1}\|Z_m\| \geq \frac{K'}{\gamma_n}\Big) \leq \frac{\gamma_n}{K'}\mathbf{E}\,(m_0^{-1}\|Z_{m_0}\|),$$

where $m_0 = m_0(n)$ is defined as $\min\{m \in \mathbb{N};\, m \geq d'\gamma_n^2\}$ and the same holds true with Z_m^* instead of Z_m. Using the following exponential inequality (e.g. (7.1) in Lemma 1 of [2])

$$\Pr\bigg(\Big\| \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \big(\delta_{Y_i} - P_n\big)\Big\| \geq \eta\bigg) \leq K_1 \exp(-K_2\eta^2), \quad \eta > 0,$$

and noting that $\gamma_n \leq d'^{-1/2} m_0(n)^{1/2}$, we obtain

$$\frac{\gamma_n}{K'}\mathbf{E}\,(m_0^{-1}\|Z_{m_0}\|) \leq \frac{\gamma_n}{K'} m_0(n)^{-1/2} K_1 \int_0^\infty \exp(-K_2 x^2)\,\mathrm{d}x \leq K'' d'^{-1/2},$$

with a positive constant K''. The same being also satisfied with $Z_{m_0}^*$ instead of Z_{m_0}, the convergence (48) follows. □

To complete the proof of Proposition 2 it remains only to prove Lemma 4. □

Lemma 4. *The probability p defined by (47) satisfies for every n large enough*

$$p \leq 24\Pr\Big(\max_{m\geq d'\gamma_n^2} m^{-1}\|Z_m\| \geq \frac{K'}{\gamma_n}\Big) + 24\Pr\Big(\max_{m\geq d'\gamma_n^2} m^{-1}\|Z_m^*\| \geq \frac{K'}{\gamma_n}\Big), \tag{49}$$

with $d' = d/2$ and $K' = K/8$.

Proof. Let us note that

$$\forall (s,t) \in T_{n,a} \setminus T_n(d), \quad |s - s^*| > \frac{d\gamma_n^2}{2n} \quad \text{or} \quad |t - t^*| > \frac{d\gamma_n^2}{2n}. \tag{50}$$

In this proof it is convenient to map for fixed n, the set of indexes $k = 1, 2, \dots, n$ into the unit circle by the transformation $s = k/n \mapsto \exp(2i\pi s)$. Thus any partial sums of Y_i's or of Y_i^* involved in the current proof may be viewed as indexed by some arc of the unit circle. Such an arc will be defined by its initial and final points and a sense of rotation from initial to final point (clockwise and anticlockwise senses will be denoted by the superscripts $\circlearrowright$ and $\circlearrowleft$ respectively). With this convention, we put for $0 < s < t \leq 1$,

$$Z^{\circlearrowleft}_{s,t} := \sum_{ns<i\leq nt} \bigl(\delta_{Y_i} - P_n\bigr), \quad Z^{\circlearrowright}_{s,t} := \sum_{\substack{0<i\leq ns \\ \text{or } nt<i\leq n}} \bigl(\delta_{Y_i} - P_n\bigr).$$

We define similarly $Z^{*\circlearrowleft}_{s,t}$ and $Z^{*\circlearrowright}_{s,t}$, substituting Y_i by Y_i^* and P_n by Q_n. Note also that $Z^{\circlearrowleft}_{s,t} = Z^{\circlearrowright}_{t,s}$.

We split $T_{n,a} \setminus T_n(d)$ in 6 subdomains corresponding to the configurations 1–6 below. To prove (49), it suffices to check that in each subdomain, we can express $S_n(s, s^*) + S_n(t^*, t)$ as a combination of at most 4 partial sums Z or Z^* having each a number of terms at least $d\gamma_n^2/2$ and at most $2n|(s,t) - \theta_n|_1$. Of course by independence and identical distribution of the Y_i's, we have for any fixed $0 < u_0 < t$, $\Pr(\|Z^{\circlearrowleft}_{u_0,t}\| > \varepsilon) = \Pr(\|Z_{t-u_0}\| > \varepsilon)$. In each configuration, we shall introduce somewhat artificially some shifts of s^* and t^* denoted by $s^\star$ and $t^\star$, whose definition shall vary with the configuration. The interest of such new endpoints is to avoid to distinguish the three subcases where $|s - s^*| \leq d\gamma_n^2/(2n)$ or $|t - t^*| \leq d\gamma_n^2/(2n)$ or both are bigger than $d\gamma_n^2/(2n)$. The points $s^\star$ and $t^\star$ will be chosen, when needed, in order that the involved arcs starting from these points contains a number of indexation points between $d\gamma_n^2/2$ and $2n|(s,t) - \theta_n|_1$. Before looking at each configuration, let us note that, due to membership of θ_n in $T_{n,a}$ for n large enough, we have $\gamma_n^2 = o(t^* - s^*)$. So from now on, let us work with n large enough to have

$$n(t^* - s^*) > 3d\gamma_n^2/2. \tag{51}$$

Configuration 1: $0 < s < t \leq s^* < t^*$, see Figure 5. Then

$$S_n(s, s^*) + S_n(t^*, t) = Z^{\circlearrowleft}_{s,t} - Z^{*\circlearrowleft}_{s^*,t^*}.$$

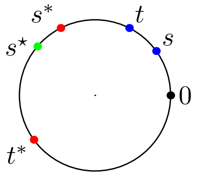

Figure 5: $0 < s < t \leq s^*$

In this configuration, $t^* - s^* \leq |(s,t) - \theta_n|_1$. So, accounting also (51), the sum $Z^{*\circlearrowleft}_{s^*,t^*}$ is suitable for our purpose. To control $Z^{\circlearrowleft}_{s,t}$, it is convenient to introduce

$s^\star := s^* + k_n/n$, where k_n is the integer part of $d\gamma_n^2/2$ and write

$$Z^{\circlearrowleft}_{s,t} = Z^{\circlearrowleft}_{s^\star,s} - Z^{\circlearrowleft}_{s^\star,t}.$$

The number of terms in each of the sums $Z^{\circlearrowleft}_{s^\star,s}$ and $Z^{\circlearrowleft}_{s^\star,t}$ is at least $d\gamma_n^2/2$ and at most $n(s^\star - s)$ which is less than $d\gamma_n^2/2 + n(s^* - s) \le 2n|(s,t) - \theta_n|_1$.

Configuration 2: $0 < s \le s^* < t \le t^*$, see Figure 6. In this case we have

$$S_n(s,s^*) + S_n(t^*,t) = Z^{\circlearrowleft}_{s^*,s} - Z^{*\circlearrowleft}_{t^*,t} = Z^{\circlearrowleft}_{s^\star,s} - Z^{\circlearrowleft}_{s^\star,s^*} - Z^{*\circlearrowleft}_{t^\star,t} + Z^{*\circlearrowleft}_{t^\star,t^*},$$

where $s^\star := s^* + k_n/n$ and $t^\star := t^* + k_n/n$.

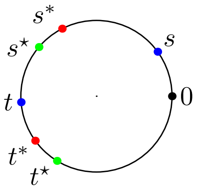

Figure 6: $0 < s \le s^* < t \le t^*$

Clearly each of these four sums has a number of terms bigger than $d\gamma_n^2/2$ and smaller than $2n|(s,t) - \theta_n|_1$.

Configuration 3: $0 < s \le s^* < t^* < t \le 1$, see Figure 7. In this case we have

$$S_n(s,s^*) + S_n(t^*,t) = Z^{\circlearrowleft}_{s^*,s} + Z^{\circlearrowleft}_{t^*,t} = Z^{\circlearrowleft}_{s^\star,s} - Z^{\circlearrowleft}_{s^\star,s^*} + Z^{\circlearrowleft}_{t^\star,t} - Z^{\circlearrowleft}_{t^\star,t^*},$$

where $s^\star := s^* + k_n/n$ and $t^\star := t^* - k_n/n$.

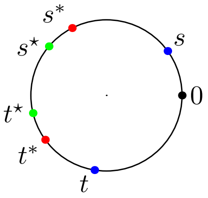

Figure 7: $0 < s \le s^* < t^* < t \le 1$

Again the four sums Z in the right hand side above have each a number of terms between $d\gamma_n^2/2$ and $2n|(s,t) - \theta_n|_1$.

Configuration 4: $s^* < s < t \le t^*$, see Figure 8. In this case we have

$$S_n(s,s^*) + S_n(t^*,t) = -Z^{*\circlearrowleft}_{s^*,s} - Z^{*\circlearrowleft}_{t^*,t} = -Z^{*\circlearrowleft}_{s^\star,s} + Z^{*\circlearrowleft}_{s^\star,s^*} - Z^{*\circlearrowleft}_{t^\star,t} + Z^{*\circlearrowleft}_{t^\star,t^*},$$

where $s^\star := s^* - k_n/n$ and $t^\star := t^* + k_n/n$.

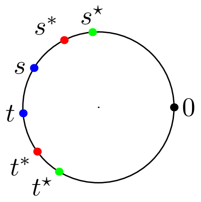

Figure 8: $s^* < s < t \le t^*$

Again the four sums Z^* in the right hand side above have each a number of terms between $d\gamma_n^2/2$ and $2n|(s,t)-\theta_n|_1$.

Configuration 5: $s^* < s \le t^* < t < 1$, see Figure 9. In this case we have

$$S_n(s,s^*) + S_n(t^*,t) = -Z^{*\circlearrowleft}_{s^*,s} + Z^{\circlearrowleft}_{t^*,t} = -Z^{*\circlearrowleft}_{s^\star,s} + Z^{*\circlearrowleft}_{s^\star,s^*} - Z^{\circlearrowleft}_{t^\star,t} + Z^{\circlearrowleft}_{t^\star,t^*},$$

where $s^\star := s^* - k_n/n$ and $t^\star := t^* - k_n/n$.

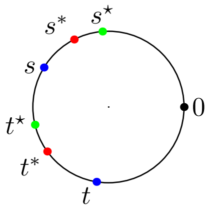

Figure 9: $s^* < s \le t^* < t < 1$

Again the four sums in the right hand side above have each a number of terms between $d\gamma_n^2/2$ and $2n|(s,t)-\theta_n|_1$.

Configuration 6: $t^* < s < t < 1$, see Figure 10. In this case we have

$$S_n(s,s^*) + S_n(t^*,t) = Z^{\circlearrowleft}_{s,t} - Z^{*\circlearrowleft}_{s^*,t^*} = Z^{\circlearrowleft}_{t^\star,t} - Z^{\circlearrowleft}_{t^\star,s} - Z^{*\circlearrowleft}_{s^*,t^*},$$

where $t^\star := t^* - k_n/n$.

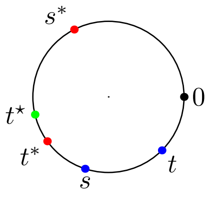

Figure 10: $t^* < s < t < 1$

The three sums in the right hand side above have each a number of terms between $d\gamma_n^2/2$ and $n|(s,t)-\theta_n|_1$. □

3 Some concluding remarks

In view of the assumption (12), theorem 1 estimates the endpoints of the epidemic interval $[ns_n^*, nt_n^*]$ only when the lenght ℓ_n^* of epidemics is of order of magnitude an for some $0 < a < 1$. This corresponds exactly to the assumption made in Dümbgen's paper, where there is only one change point θ_n converging to $\theta \in (0, 1)$.

In recent papers [3, 4] the authors proposed test statistics based on some Hölder norms to detect very short epidemics of lenght $\ln^\gamma n$, $\gamma > 1$. The estimation of the interval of epidemic in this more general setting is treated in the paper [5] which should appear elsewhere. In this paper one estimate the pair (s^*, h^*) instead of (s^*, t^*). The convergence rate obtained is the same as in theorem 1, under a condition which in the simpler case where the semi norm N_n is non random writes

$$\frac{\ln n}{\ell^* N_n(Q_n - P_n)^2} \xrightarrow[n\to\infty]{} 0.$$

This is a generalization of the second condition in our assumption (11), where ℓ_n^* is of the same order of magnitude as n by (12). The method exposed in the current paper could be refined to obtain similar result relaxing assumption (12), using the fact that in Lemma 3 ii), the constant C is of the order of $w(h_n^*)$.

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